YOUNG-JUCYS-MURPHY ELEMENTS

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Abstract. In this lecture, we will be considering the branching multigraph of irreducible representations of the S_n, although the morals of the arguments are applicable to more general cases. We will liberally apply criteria about the centralizer of a subrepresentation to show that that the restriction of an irreducible representation to a subrepresentation has simple multiplicity, which will show that the branching graph of irreducible representations of S_n is in fact simple. We will then define the Young-Jucys-Murphy elements in $\mathbb{C}[S_n]$, show that they in fact generate the Gelfand-Tsetlin algebra, and see how they relate to the GZ-basis.

N.B. We will assume that all vector spaces are finite-dimensional over $\mathbb{C}$.

Contents

1. Set-up

Recall the following:

Let $\{1\} = G_0 < G_1 < G_2 < \cdots$ be a chain of finite groups, and denote $\hat{G} = \{\text{irreps of G}\}$ (irreducible = no G-invariant subspaces).

For each $\rho \in \widehat{G}_n$, V^ρ decomposes into a direct sum of irreps $\mu \in \widehat{G}_{n-1}$ with multiplicities $m_\mu = \dim \operatorname{Hom}_G(V^\mu, V^\rho)$.

$$V^\rho = \bigoplus_{\mu \in \widehat{G}_{n-1}} (V^\mu)^{m_\mu}.$$

The *branching graph* is the directed multigraph whose vertices are elements of $\bigsqcup_{k\geq 0} \widehat{G}_k$, with $\hat{G}_n$ called the nth *level*.

Two vertices $\mu \in \widehat{G}_{n-1}$ and $\rho \in \hat{G}_n$ are connected by k dirrected edges if $k = \dim \operatorname{Hom}_{G_{n-1}}(V^\mu, V^\rho)$.

Write $\mu \nearrow \rho$ if μ and ρ are connected, i.e if V^μ is a factor in the decomposition of V^ρ.

Call the branching graph *simple* if all multiplicities are either 0 or 1, in which case

$$V^\rho = \bigoplus_T V_t$$

$V_t := \mathbb{C}$, $t \in T = \{$ increasing paths $t = \{\rho_0 \nearrow \rho_1 \nearrow \cdots \nearrow \rho_n = \rho\}$, $\rho_i \in \widehat{G}_i$ $\}$

Choose units $v_t \in V_t\ \forall t \in T$, then the $GZ-basis$ of $V^\rho := \{v_t : t \in T\}$

Remark 1.1. What this looks like is choosing some generator of $V^{\{1\}}$, and tracking its image under each path in V^ρ.

Definition 1.2. The *Gelfand-Tsetlin algebra* GZ_n is the algebra generated by the centers $Z_1 \subset \mathbb{C}[G_1], ..., Z_n \subset \mathbb{C}[G_n]$; that is, $GZ_n =< Z_1, \cdots, Z_n >$

Proposition 1.3. *GZ_n is the maximal commutative subalgebra in $\mathbb{C}[G_n]$ when the branching graph is simple, and consists of operators that are diagonal in the GZ-basis.*

Remark 1.4. We will eventually be looking at how GZ_n acts on each irrep.

We also recall the following lemma which will be very handy:

Lemma 1.5. *("Criteria"): Let M be a semisimple finite-dimensional $\mathbb{C}$-algebra, $N \subset M$ a subalgebra. The centralizer $Z_N(M)$ is commutative if and only if, for any $\rho \in \widehat{M}$, the restriction $\mathrm{Res}^M_N V^\rho$ of an irrep of M to N has simple multiplicities.*

As well, keep in mind that irreps (i.e. irreducible representations) of a finite group $G \longleftrightarrow \mathbb{C}[G]-modules$.

N.B. From now on, take $G_n = S_n$.

2. Assorted Lemmas, Tidbits, and Facts

We will be looking at the branching graph of irreps of $\mathbb{C}[S_n]$ to study the branching graph of S_n. We can do this because we have a bijection:

$$Hom_{groups}(G, GL(n, \mathbb{C})) \cong Hom_{\substack{unital\\assoc.\\algebras}} (\mathbb{C}[G], Mat_{nxn}(n, \mathbb{C}))$$

To see why this is, if you have any homomorphism on the elements of G, you can extend it to $\mathbb{C}[G]$, and likewise any homomorphism of $\mathbb{C}[G]$ can be restricted to the basis G. We also need to think about invertibility, but will hand-wave that for now.
While S_n has a trivial centre, the group algebra$\mathbb{C}[S_n]$ does not, so we can get more information by working with it.

Note the following fact from elementary algebra:

Fact 2.1. Conjugation preserves cycle type, and if $\sigma = (i_1, \cdots, i_k)$, then

$$\tau\sigma\tau^{-1} = (\tau(i_1), \cdots, \tau(i_k))$$

Proof. Consult any algebra textbook ever written. □

As we will be proving results about the centres and centralizers of $\mathbb{C}[S_n]$, let's think about what the centre $Z_n := Z(\mathbb{C}[S_n]) = \{\ z \in \mathbb{C}[S_n]) : \text{zy} = \text{yz}\ \forall y \in \mathbb{C}[S_n])\ \}$ looks like:

Proposition 2.2. *Let $z \in Z_n$, $z = \sum_{g \in S_n} c_g g$. For any $g \in S_n$, $h \in S_n$, $hzh^{-1} = z$. Since conjugation will permute the g_i's within the conjugacy class $[g]$ (which consists of all permutations of a particular cycle type), we must have that c_{g_i}'s are all the same, i.e.*

$$z = \sum_{[g]} c_{[g]} \sum_{g_i \in [g]} g_i$$

So, we can describe Z_n as:

$$Z_n = \{Z_\lambda | \lambda \dashv n\}$$

where

$$Z_\lambda = \sum_{\substack{\sigma \in S_n \\ w/cycletype\lambda}} \sigma$$

Lemma 2.3. *(Lemma 1) Every $g \in S_n$ is conjugate to $g^{-1} \in S_n$, i.e. $\exists h \in S_n$ s.t. $g^{-1} = hgh^{-1}$. Moreover, we can find such an $h \in S_{n-1}$.*

Proof. Clearly every permutation $\in S_n$ is conjugate to its inverse, since if $\sigma = (i_1, ..., i_n)$, then $\sigma^{-1} = (i_n, i_{n-1}..., i_1)$, $\rho\sigma\rho^{-1} = (\rho(i_1), ...\rho(i_n))$, choose ρ s.t. $\rho(i_1, ..., i_n) = (i_n, i_{n-1}, ..., i_1)$.

For $g \in S_n$, let $g' \in S_{n-1}$ be the induced permutation in S_{n-1}. Take $h \in S_{n-1}$ that conjugates g' and g'^{-1}, so $g'^{-1} = hg'h^{-1}$. Then h has the fixed point n, so extended to S_n, it satisfies $g^{-1} = hgh^{-1}$. $\square$

Before the next lemma, it's time for another definition:

Definition 2.4. An *involution algebra*, or a $* - algebra$, is an algebra A with a map (called an involution) $* : A \to A$ satisfying:

(1) $(a^*)^* = a$
(2) $(ab)^* = b^* a^*$
(3) $(\lambda a + b)^* = \bar{\lambda} a^* + b^*$

Call a^* the *conjugate* or *adjoint* of a.

Example 2.5. $\mathbb{C}$ is a $*$-algebra over $\mathbb{R}$ with $*$ = conjugation.

Definition 2.6. In an $* - algebra\ A$, call $a \in A$ *normal* if a commutes with its conjugate, i.e. $aa^* = a^*a$, and call $a\ self - adjoint$ if $a = a^*$.

Lemma 2.7. *(Lemma 2) Let A be an $* - algebra$ over $\mathbb{C}$. Then*

(1) A is commutative $\iff$ all of its elements are normal.
(2) If every real element is self-adjoint, then A commutative.

Proof. (1) $\Rightarrow$: Trivial
$\Leftarrow$: Suppose $aa^* = a^*a\ \forall a \in A$, and denote $A_{sa} = \{a \in A : a \text{ self-adjoint}\}$. A can be decomposed as $A = A_{sa} + iA_{sa}$ (by properties of $*$). If $a, b \in A_{sa}$, then $a = a^*$ and $b = b^*$, so $(a+ib)^* = a^* + \bar{i}b^* = a - ib$. But $(a + ib)$ normal $\Rightarrow (a + ib)(a - ib) = (a - ib)(a + ib) \Rightarrow ab = ba$, i.e. a and b commute. Hence A commutative.

(2) Let $A_{\mathbb{R}}$ be the real subalgebra of A, i.e. $A = \mathbb{C} \bigotimes A_{\mathbb{R}}$, and assume all $a \in A$ are self-conjugate (i.e. $a = a^* \forall A \in A$). Then $a, b \in A_{\mathbb{R}} \Rightarrow ab = (ab)^* = b^*a^* = ba$, so $A_{\mathbb{R}}$ commutative, but then so is $A = \mathbb{C} \bigotimes A_{\mathbb{R}}$.

□

3. Centralizers and Centres

Theorem 3.1. *The centralizer $Z_{\mathbb{C}[S_{n-1}]}(\mathbb{C}[S_n]) =: Z_{n-1}(n)$ of $\mathbb{C}[S_{n-1}]$ in $\mathbb{C}[S_n]$ is commutative.*

Proof. By lemma 2, we have reduced this problem to checking that every real element of the centralizer $Z_{n-1}(n) \subset \mathbb{C}[S_n]$ is self-adjoint.

Let $\{ g_i \colon g_i \in S_n \}$ be a basis for $\mathbb{C}[S_n]$, $f = \sum_i c_i g_i \in Z_{n-1}(n)$ c_i 's $\in \mathbb{R}$.

What does self-adjoint look like in $\mathbb{C}[S_n]$? The adjoint of g, viewing g as a function $g : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$, can also be categorized as the function $g^* : \mathbb{C}[S_n] \to \mathbb{C}[S_n]$ such that$< ga, b >=< a, g^*b > \forall a, b \in \mathbb{C}[S_n]$. Take the inner product $< g_i, g_j >= \delta_{ij}$. Then $< g_i, g_j >=< gg_i, gg_j >$, and by def. of adjoint we have $< gg_1, g_2 >=< g_1, g^*g_2 >=< g_1, g^{-1}g_2 >\Rightarrow g^* = g^{-1}$.

So, $(\sum_i c_i g_i)^* = \sum_i c_i * g_i^{-1} = \sum_i c_i g_i^{-1}$. $f \in Z_{n-1}(n)$ commutes with all $h \in S_{n-1}$, so since the expression for f is unique, this expression is invariant under conjugation by $h \in S_{n-1}$, $f \to hfh^{-1} = f$. By lemma 1, we can choose h_i s.t. $h_i g_i h_i^{-1} = g_i^{-1}$, so $\sum_i c_i g_i \mapsto \sum_i c_i g_i^{-1}$. But since $hfh^{-1} = f$, that means that $c_i g_i$ a summand $\Rightarrow c_i g_i^{-1}$ a summand, so $f^* = f$. □

In light of this, we have the following:

Theorem 3.2. *A a finite-dimensional $*$-algebra over $\mathbb{R}$, B a $*$-subalgebra. Let $G = \{ g_i \}$ be a basis of A that is closed under $*$, and s.t. $\forall i, \exists$ orthogonal $b_i \in B$ ($b^* = b^{-1}$) s.t. $b_i g_i b_i^* = g_i^*$. Then $Z_B(A)$ is commutative, and hence the restriction of any irrep from A to B has simple multiplicity.*

In particular, if A is a group algebra of a finite group G and B of $H \leq G$, where A has basis G, then $\forall g \in G, \exists h \in H$, $g' \in G$ s.t. $h^{-1}g'h = g$, and if we can take $g' = g$, then $Z_B(A)$ is commutative, and hence the restriction of any irrep from A to B has simple multiplicity.

Remark 3.3. The proof we just gave for the simplicity of the spectrum (i.e. the branching of the multigraph) didn't require any knowledge of the representations themselves, only elementary algebraic properties of the group.

4. Young-Jucys-Murphy Elements

Remark 4.1. We can also look at the branching through analyzing the centralizers of the group algebras, so we're going to develop a more detailed description of the centralizer $Z_{n-1}(n)$ and its relation to GZ_n.

Definition 4.2. The $Young-Jucys-Murphy$ elements (henceforth abbreviated as $YJM-elements$) are

$$X_i = (1i) + (2i) + ... + ((i-1)i) \in \mathbb{C}[S_n]$$

Remark 4.3. $X_i = \sum$ (transpositions in S_i) - $\sum$ (transpositions in S_{i-1}), so X_i is the difference of an element in $Z(i)$ and $Z(i-1)$, so $X_i \in GZ_n$; in particular, the X_i's commute.

Proposition 4.4. *$X_k \notin Z_k$ for any k.*

Proof. $Z_k = span\{\sum_{\substack{\sigma\in S_n \\ cycletype\lambda}} \sigma|\lambda$ a partition of k $\}$, so for cycle type λ = 2,1,1,...,1, any element $\in Z_n$ consisting of 2-cycles must be expressed in terms of all the 2-cycles $\in S_n$. □

Theorem 4.5. *The centre $Z_n \subset \mathbb{C}[S_n]$ is a subalgebra of the one generated by the centre $Z_{n-1} \subset \mathbb{C}[S_{n-1}]$ and X_n:*

$$Z_n \subset < Z_{n-1}, X_n >$$

Proof. (Sketch)

(1) Show all classes of one-cycle type permutations lie in $< Z_{n-1}, X_n >$:
$X_n = \sum_{i=1}^{n-1}(in) = \sum_{i\neq j;i,j=1}^{n}(ij)$ - $\sum_{i\neq j;i,j=1}^{n-1}(ij)$, where the first sum $\in < Z_{n-1}, X_n >$ and the second $\in Z_n-1$. Note that $X_n^2 = \sum_{i\neq j;i,j=1}^{n}(in)(jn) = \sum_{i\neq j;i,j=1}^{n-1}(ijn)+1$, so $X_n^2+\sum_{i\neq j\neq k;i,j,k=1}^{n-1}(ijk) = \sum_{i\neq j\neq k;i,j,k=1}^{n}(ijk) \in < Z_{n-1}, X_n >$.
In a similar fashion, we can show that the sum $X_n^{k-1}+\sum_{i_1\neq i_2\neq \ldots \neq i_k;i_1,\ldots,i_k=1}^{n}(i_1 \cdots i_k) \in < Z_{n-1}, X_n >$, so that all one-cycle type classes in Z_n lie in $< Z_{n-1}, X_n >$.

(2) Apply the general theorem that Z_n is generated by the classes of one-cycle type permutations (i.e. these permutations are the multiplicative generators) to conclude that $Z_n \subset < Z_{n-1}, X_n >$.

□

Corollary 4.6. *The algebra GZ_n is generated by the $YJM-elements$, i.e.*

$$GZ_n = < X_1, ...X_n >$$

Proof. By definition, $GZ_n = < Z_1, ..., Z_n >$. We will proceed by induction:
$GZ_2 = \mathbb{C}[S_2] = < X_1 = 0, X_2 >= \mathbb{C}$. Now, assume by induction that $GZ_{n-1} = < X_1, ..., X_{n-1} >$.
NTS $GZ_n = < GZ_{n-1}, X_n >$.
$\supset$: Clearly $GZ_n \supset < GZ_{n-1}, X_n >$ since $GZ_{n-1} \subset GZ_n$ and $X_n \in Z_n$.
$\subset$: By the previous theorem, $Z_n \subset < Z_{n-1}, X_n > \subset < GZ_{n-1}, X_n >$. □

Proposition 4.7. *$X_k \notin Z_k$ for any k.*

Proof. $Z_k = span\{\sum_{\substack{\sigma\in S_n \\ cycletype\lambda}} \sigma|\lambda \dashv$ k $\}$, so for cycle type λ = 2,1,1,...,1, any element $\in Z_n$ consisting of 2-cycles must be expressed in terms of all the 2-cycles $\in S_n$. □

Theorem 4.8. *The centralizer $Z_{n-1}(n)$ is generated by the centre $Z_{n-1} \subset \mathbb{C}[S_n]$ and the YJM-element X_n:*

$$Z_{n-1}(n) = < Z_{n-1}, X_n >$$

Proof. Let's consider a basis for $Z_{n-1}(n)$. This will be the union of the basis of Z_{n-1}, along with classes of the form:

$$\{\sum (i_{1,1} \cdots i_{1,k-1}n)(i_{2,1} \cdots i_{2,k_2}) \cdots (i_{j,1} \cdots i_{n,k_j})|k_1, ..., k_j \dashv n\}$$

where $i_{j,k} \in$ { 1, ..., n-1 } and the cycle lengths $k_1, ..., k_j$ over possible partitions of n. This is because conjugating by any element $h \in Z_{n-1}$ must fix any $z \in Z_{n-1}(n)$, and so

$$\begin{aligned} h(\sum (i_{1,1} \cdots i_{1,k_1-1}n)(i_{2,1} \cdots i_{2,k_2}) \cdots (i_{n,1} \cdots i_{n,k_n}))h^{-1} \\ = \sum (h(i_{1,1}) \cdots h(i_{1,k-1})h(n))(h(i_{2,1}) \cdots h(i_{2,k_2})) \cdots (h(i_{n,1}) \cdots h(i_{n,k_n})) \\ = \sum (h(i_{1,1}) \cdots h(i_{1,k-1})n)(h(i_{2,1}) \cdots h(i_{2,k_2})) \cdots (h(i_{n,1}) \cdots h(i_{n,k_n})) \\ = \sum (i_{1,1} \cdots i_{1,k-1}n)(i_{2,1} \cdots i_{2,k_2}) \cdots (i_{n,1} \cdots i_{n,k_n}) \end{aligned}$$

Since n must stay in the cycle of length k_1 and h permutes the rest of the elements. As in the proof showing that $Z_n \subset < Z_{n-1}, X_n >$, we can take the sum of these classes with the corresponding classes:

$$\{\sum (i_{1,1} \cdots i_{1,k})(i_{2,1} \cdots i_{2,k_2}) \cdots (i_{j,1} \cdots i_{n,k_j}) | k_1, ..., k_j \dashv n\}$$

from Z_{n-1} to obtain classes in Z_n, which shows that a basis of Z_{n-1} can be obtained via a linear combination of elements in the bases of Z_{n-1} and Z_n:

$$Z_{n-1}(n) \subset < Z_{n-1}, Z_n >$$

But we already proved that $Z_n \subset < Z_{n-1}, X_n >$, so

$$Z_{n-1}(n) \subset < Z_{n-1}, X_n >$$

Conversely, $Z_{n-1} \subset Z_{n-1}(n)$ and X_n commutes with all elements $\in Z_{n-1}$, so we also have

$$Z_{n-1}(n) \supset < Z_{n-1}, X_n >$$

□

5. Simplicity of Branching

Theorem 5.1. *(Main) The branching of the chain $\mathbb{C}[S_1] \subset \cdots \subset \mathbb{C}[S_n]$ is simple, hence the same is true for $S_1 < ... < S_n$.*

Proof. $Z_{n-1}(n) \subset < Z_{n-1}, X_n > \subset GZ_n$ commutative $\Rightarrow$ any restriction from $\mathbb{C}[S_n]$ to $\mathbb{C}[S_{n-1}]$ is simple by "Criteria". □

Corollary 5.2. *GZ_n is the maximal commutative subalgebra of $\mathbb{C}[S_n]$, and in each irrep of S_n, the GZ-basis is determined up to scalar factors (as a consequence of Schur's lemma).*

Definition 5.3. The union of GZ-bases of irreps $\in \hat{S_n}$ is called the *Young basis.*

Proposition 5.4. *Let v be a vector in the Young basis of some irrep, and denote the weight of v:*

$$\alpha(v) = (a_1, ..., a_n) \in \mathbb{C}^n$$

be the eigenvalues of $X_1, ..., X_n$ on v. Denote the spectrum of YJM-elements as

$$Spec(n) = \{\alpha(v) | v \in Youngbasis\}$$

Then $\alpha \in Spec(n)$ determines v up to scalar multiplication, and we can easily see that $|Spec(n)| = \sum_{\lambda \in \hat{S_n}} dim(\lambda) = dim(GZ_n)$ i.e. $dim(GZ_n)$ = the sum of pairwise non-isomorphic irreps

Proposition 5.5. *We also have a bijection between $Spec(n)$ and the set of paths in the branching graph, and a natural equivalence relation on Spec(n): if v_α and v_β the corresponding vectors in the Young basis corresponding to weights α and $\beta \in Spec(n)$, then α β if the paths T_α and T_β have the same end, i.e. v_α and v_β are in the same irrep, so that $|Spec(n)/ \ | = |\hat{S_n}|$*

Remark 5.6. Later, we will see how these eigenvectors can be found from combinatorial data from Young tableaux.

6. Takeaways

(1) YJM-elements $X_n = (1n) + ... + ((n-1)n)$
(2) $GZ_n =< Z_1, ..., Z_n >=< X_1, ..., X_n >$
(3) The branching of the multigraph of $\mathbb{C}[S_n]$ and therefore that of S_n is simple
(4) The spectrum of the YJM-elements determines the branching of the graph

References

[1] Vershik and Okounkov A New Approach to the Representation Theory of Symmetric Groups, II ftp://ftp.ams.org/pub/tex/doc/amsmath/short-math-guide.pdf